

Critical Study on the Absorbing Phase Transition in a Four-State Predator-Prey Model in One Dimension

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Abstract. In a recent letter [Chatterjee R, Mohanty P K, and Basu A, 2011 *J. Stat. Mech.* L05001], it was claimed that a four-state predator-prey (4SPP) model exhibits critical behavior distinct from that of directed percolation (DP). In this Letter, we show that the 4SPP in fact belongs to the DP universality class.

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Absorbing phase transitions have been studied extensively as prototypical emergent phenomena of nonequilibrium systems and several universality classes have been identified (for a review, see, e. g., Refs. [1, 2, 3, 4]). Among them, the directed percolation (DP) universality class has been studied thoroughly both numerically and analytically. As a result, a so-called ‘DP-conjecture’ [5, 6] has emerged since early 1980’s, asserting that a model should belong to the DP class, if the model exhibits a phase transition from an active, fluctuating phase into a *unique* absorbing state, if the phase transition should be characterized by a one-component order parameter, if no quenched disorder is involved, if dynamic rules of the model are short-ranged processes, and if the model has neither novel symmetry nor conservation. The validity of the conjecture has been supported by numerical evidences. Although the theory suggests that the DP class is robust, an experimental realization of the DP class was successful only recently [7].

Theoretically, it has been questioned if each condition for the conjecture is necessary. Indeed, some of conditions are shown to be necessary. It is by now firmly established that conditions of no quenched disorder and of short-range processes are necessary for the DP critical behavior (for a review of non-DP critical behavior by quenched disorder or by long-range interactions, see, e. g., Ref. [3]). On the other hand, it turns out that certain models that violate some of the above mentioned conditions still belong to the DP class. At first, the condition for a *unique* absorbing state is shown to be not necessary. A prototypical model is the pair contact process (PCP) [8] which has infinitely many absorbing states (IMAS) but belongs to the DP class. Second, the condition of the absence of symmetry or conservation should be applied with care. For instance, a parity conservation sometimes changes the universality class as happens in branching annihilating random walks with even number of offspring [9, 10], but sometimes it has no role as in a model of diffusing particles with reaction dynamics, $2A \rightarrow 0$ and $3A \rightarrow 5A$ [11, 12].

More notably, the condition of a one-component order parameter is rather tricky because this condition requires coarse-graining of microscopic models. In particular, the coarse-graining should result in reggeon field theory [13]. A question in this direction arises if we have a systematic prescription to conclude that an absorbing phase transition of a certain microscopic model is characterized by a one-component order parameter field theory. A naive answer would be the number of species in a microscopic model. Here, by species is meant a group of particles whose dynamic rules are same irrespective of position and time. However, there are counter-examples against this naive answer. At first, certain models such as the Ziff-Gulari-Barshad model [14] also exhibit the DP criticality, although there are two different species in the model. Second, to make things even more complicated, certain single-species microscopic model may have multi-component order parameters in field theory. To explain the meaning of the above sentence in a concrete way, let us consider single-species models with “hybrid” rules such as $nA \rightarrow 0$ and $mA \rightarrow (m+1)A$ ($n, m \geq 1$) of diffusing A particles [11], which show different critical behavior depending on the values of n and m . If $m > n$, this

model belongs to the DP class [11]. However, if $m = n$, the universality class to which this model belongs differs with n . For the case of $m = n = 1$, it still belongs to the DP class. For the case of $m = n = 2$ which is also known as the pair contact process with diffusion (PCPD) (for a review see, e. g., Refs. [15, 16]) and for the case of $m = n = 3$ which is also known as the triplet-contact process with diffusion (TCPD) [17], these models do not share criticality with DP at least in dimensions higher than 1. Regarding the number of order parameter components, it was argued by studying the PCPD with biased diffusion [18, 19] that the field theory of the PCPD should have at least two different order parameter components, which cannot be reduced to a local field theory with a single component in the asymptotic regime. Failure of a single-component field theory for the PCPD [20] also suggests the same conclusion arrived at in Ref. [18]. To summarize, the condition of single component order parameter cannot be answered by simply looking at dynamic rules of a model.

In this context, the conclusion in the recent Letter by Chatterjee et al. [21] is worth deep investigation. In that Letter, it was asserted by numerical studies that a four-state predator-prey (4SPP) model does not belong to the DP class although the property of the model does not seem to deviate much from the conditions of the DP-conjecture. If the claim by Chatterjee et al. [21] were right, a theory should be developed why the 4SPP model does not belong to the DP class, which would deepen our understanding of the DP class. In this Letter, we will study the 4SPP model in detail.

Unfortunately, however, there is an apparent fallacy in Ref. [21]. The 4SPP model without diffusion was claimed to have only two absorbing states one of which is dynamically unstable. In fact, however, there are infinitely many absorbing states (see below). Because of this mistake, Chatterjee et al. [21] chose a wrong order parameter, which eventually led them to locate a wrong critical point. In this Letter, we will show that the critical point in Ref. [21] is actually underestimated and at the right critical point, the 4SPP model exhibits the DP scaling.

For completeness, we will begin with explaining the 4SPP model. In the 4SPP model, there are two species; prey (called A) and predator (called B). At most one individual from each species can reside at any lattice point, but two different species can stay together at the same site. Hence, each site can have one of four possible states; empty, A alone occupied, B alone occupied, and both A and B occupied. A can spontaneously generate an offspring to its nearest neighbor, but B cannot branch its offspring without consuming A . However, unlike A species, predators also compete with each other for space and if two predators are located in a row, both of them can be killed together and removed from the system. In this Letter, we restrict ourselves to a one-dimensional lattice system with size L . Periodic boundary conditions are always assumed.

To describe the dynamic rules of the model in detail, we introduce a state variable s_i at site i ($1 \leq i \leq L$) such that $s_i = s_i^A + 2s_i^B$, where s_i^A (s_i^B) is the number of A (B) species at site i . Obviously, s_i can take one of four possible values, 0, 1, 2, and 3. We also assign a bond variable b_i to a bond connecting i -th and $(i + 1)$ -th sites such that

State	$\circ \circ$ $\circ \circ$	$\circ \textcircled{A}$ $\circ \circ$	$\circ \circ$ $\circ \textcircled{B}$	$\circ \textcircled{A}$ $\circ \textcircled{B}$	$\textcircled{A} \circ$ $\circ \circ$	$\textcircled{A} \textcircled{A}$ $\circ \circ$	$\textcircled{A} \circ$ $\circ \textcircled{B}$	$\textcircled{A} \textcircled{A}$ $\circ \textcircled{B}$
b_i	0	1	2	3	4	5	6	7
State	$\circ \circ$ $\textcircled{B} \circ$	$\circ \textcircled{A}$ $\textcircled{B} \circ$	$\circ \circ$ $\textcircled{B} \textcircled{B}$	$\circ \textcircled{A}$ $\textcircled{B} \textcircled{B}$	$\textcircled{A} \circ$ $\textcircled{B} \circ$	$\textcircled{A} \textcircled{A}$ $\textcircled{B} \circ$	$\textcircled{A} \circ$ $\textcircled{B} \textcircled{B}$	$\textcircled{A} \textcircled{A}$ $\textcircled{B} \textcircled{B}$
b_i	8	9	10	11	12	13	14	15

Figure 1. The connection between the bond variable b_i and the (local) state at sites i (circles in left columns) and $i + 1$ (circles in right columns). Circles in the top (below) rows indicate whether A (B) species is present or not. For example, the state corresponding to $b_i = 11$ is that at site i only B is occupied and at site $i + 1$, both A and B are occupied.

$b_i = 4s_i + s_{i+1}$ (recall the periodic boundary conditions). The connection between a local configuration and a bond variable is depicted in Figure 1.

At first, we assume that both A and B are not diffusive. As in Ref. [21], branching occurs in a biased way. In this case, the bond variables changes by the following stochastic rules:

$$\begin{aligned}
 &4 \rightarrow 5 \text{ with rate } p, & 6 \rightarrow 7 \text{ with rate } p, \\
 &9 \rightarrow 10 \text{ with rate } r, & 10 \rightarrow 0 \text{ with rate } q, \\
 &11 \rightarrow 1 \text{ with rate } q, & 12 \rightarrow 13 \text{ with rate } p, \\
 &13 \rightarrow 14 \text{ with rate } r, & 14 \rightarrow \begin{cases} 15 \text{ with rate } p, \\ 4 \text{ with rate } q \end{cases} \\
 &15 \rightarrow 5 \text{ with rate } q,
 \end{aligned} \tag{1}$$

where numbers on the left (right) hand side of the arrows signifies the bond variables before (after) a change with corresponding transition rates. p is the transition rate of A branching event, q is the transition rate of pair annihilation of B species by competition, and r is the transition rate of B branching event by consuming one A . Since a configuration change can occur if a bond variable is an element of the following set

$$S = \{4, 6, 9, 10, 11, 12, 13, 14, 15\}, \tag{2}$$

we will refer to a bond with $b_i \in S$ as an active bond.

In simulations, we keep a list of active bonds at each time. We choose a time-increment dt such a way that

$$0 < dt \leq \frac{1}{\max(p + q, r)}. \tag{3}$$

Assume that there are $N(t)$ active bonds at time t . Among $N(t)$ active bonds, we choose one at random. If $b_i = 4, 6$, or 12 , b_i increases by 1 with probability $p dt$ (branching of A species). In case this change happens, b_{i+1} also increases by 4. If $b_i = 9$ or 13 , b_i

increases by 1 with probability $r dt$ (branching of B species by consuming one A). In case this change happens, b_{i+1} also increases by 4. If $b_i = 10, 11$, or 15 , b_i decreases by 10 with probability $q dt$ (pair annihilation of two predators). In case this change happens, b_{i+1} (b_{i-1}) also decreases by 8 (2). If $b_i = 14$, b_i increases by 1 with probability $p dt$ (branching of A species) or decreases by 10 with probability $q dt$ (pair annihilation of two predators). If one of these two events happens, the bond variables of its neighbors also changes accordingly. After an attempt of a configuration change, time increases by $dt/N(t)$ and the list of active bonds is to be updated. We terminate the above procedure if time exceeds the predetermined maximum observation time or the system does not have any active bonds. In the following, we choose the transition rates in such a way that dt can be chosen to be 1. In all simulations, the initial condition is that $s_i = 3$ for all i .

Clearly, a configuration without any active bonds is absorbing in the sense that no further configuration change can occur. Hence, unlike the claim in Ref. [21], the 4SPP model without diffusion has infinitely many absorbing states (any configuration with only isolated predators is absorbing). Because Chatterjee et al. [21] ignored the absorbing nature of isolated predators, the density of predators is wrongfully chosen as an order parameter. Since the 4SPP is an IMAS model and the density of predators need not be zero in the absorbing state, the critical point reported in Ref. [21] is not the correct one. Actually, the density of active bonds defined as

$$\rho(t) \equiv \frac{N(t)}{L}, \quad (4)$$

where $N(t)$ is the number of active bonds (see above) at time t and L is the system size, can characterize correctly the onset of phase transition.

At first, we study how $\rho(t)t^\delta$ with $\delta \simeq 0.1594$ (DP exponent [22]) behaves near criticality. To compare our results with those in Ref. [21], we fix $r = 0.9$ and $q = 0.02$ and the criticality is found by varying p . We simulated the system with size $L = 2^{20}$ and the number of independent runs, for instance, for $p = 0.153\ 81$ is 400. Up to the observation time $t_{\max} \approx 3 \times 10^7$, all simulated systems did not fall into one of absorbing states, which minimally guarantees that the finite size effect is not operative. Note that the system size and the observation time in this Letter are much larger than those in Ref. [21]. In Figure 2(a), semi-logarithmic plots of the density behavior for various p 's are depicted. From $t = 10^5$, one can see that the curve for $p = 0.153\ 81$ becomes almost constant, and the curve for $p = 0.153\ 80$ ($0.153\ 82$) veers down (up), which indicates that the critical point is $p_c = 0.153\ 81(1)$ with the number in parentheses to be the error of the last digit.

To be more systematic, we also study the effective exponent defined as (b is a certain fixed constant)

$$-\delta(t) \equiv \frac{\ln \rho(t)/\rho(t/b)}{\ln b}. \quad (5)$$

In the active phase the effective exponent should veer up because $\rho(t)$ should saturate to a finite number. In the absorbing phase, on the other hand, the effective density should

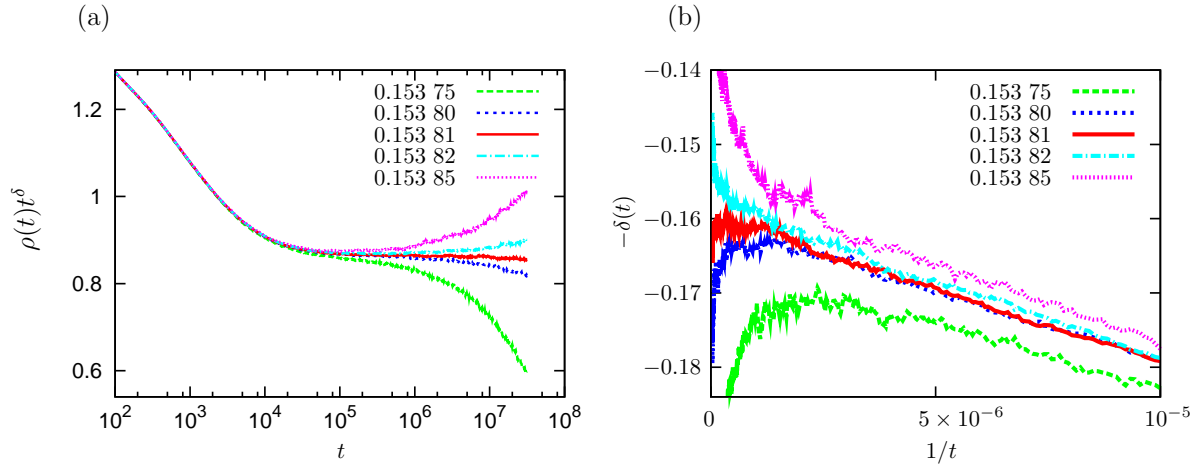


Figure 2. (a) Semi-logarithmic plots of $\rho(t)t^\delta$ vs t with DP critical exponent $\delta = 0.1594$ for $p = 0.153\,75, 0.153\,80, 0.153\,81, 0.153\,82$, and $0.153\,85$ with $q = 0.02$ and $r = 0.9$. (b) Plots of effective exponents $-\delta(t)$ as a function of $1/t$ for the simulation results in (a). Since the effective exponent veers down (up) in the absorbing (active) phase, the critical point is estimated as $p_c = 0.153\,81(1)$. Moreover, the effective exponent approaches to the DP critical exponent 0.1594 with linear behavior in $1/t$.

veer down because $\rho(t)$ decays exponentially (or at least faster than the critical decay). And only at criticality, $\delta(t)$ will approach to the critical exponent δ as $t \rightarrow \infty$. Note that if the leading corrections to scaling takes the $1/t$ form, that is, if $\rho(t) \approx t^{-\delta}(1 + a/t)$ in the asymptotic regime with a constant a , which is the case in many models belonging to the DP class, $\delta(t)$ at criticality approaches to δ linearly if $\delta(t)$ is drawn as a function of $1/t$.

We analyze the effective exponent using the same data in Figure 2(a) with $b = 10$. The resulting plots of the effective exponents are shown in Figure 2(b). Again we conclude that the critical point is $0.153\,81(1)$ and the effective exponent approaches to the DP critical exponent 0.1594 linearly in $1/t$.

To make our point stronger, we also study the finite size scaling. To this end, we simulated the system at criticality $p = 0.153\,81$ with varying system sizes from $L = 2^8$ to $L = 2^{13}$. Figure 3(a) shows the density decay behavior for different system size. Since the order parameter takes the form in the scaling regime

$$\rho(t) = F(t/L^z)L^{-\beta/\nu_\perp}, \quad (6)$$

where z , ν_\perp , and β are critical exponents and F is a scaling function, plots of $\rho(t)L^{\beta/\nu_\perp}$ vs t/L^z for different L should collapse into a single curve in the scaling regime. Figure 3(b) shows a nice scaling collapse, using $z = 1.58$ and $\beta/\nu_\perp = 0.252$ (DP exponents [22]), which again confirms that the 4SPP belongs to the DP class.

Up to now, we assume that both A and B species are not diffusive and we show that the 4SPP model with non-diffusive species does belong to the DP class. Now we will study if diffusive motion can affect the universal behavior. In Ref. [21], though no explicit numerical results are shown, the 4SPP model with diffusion is also claimed not to

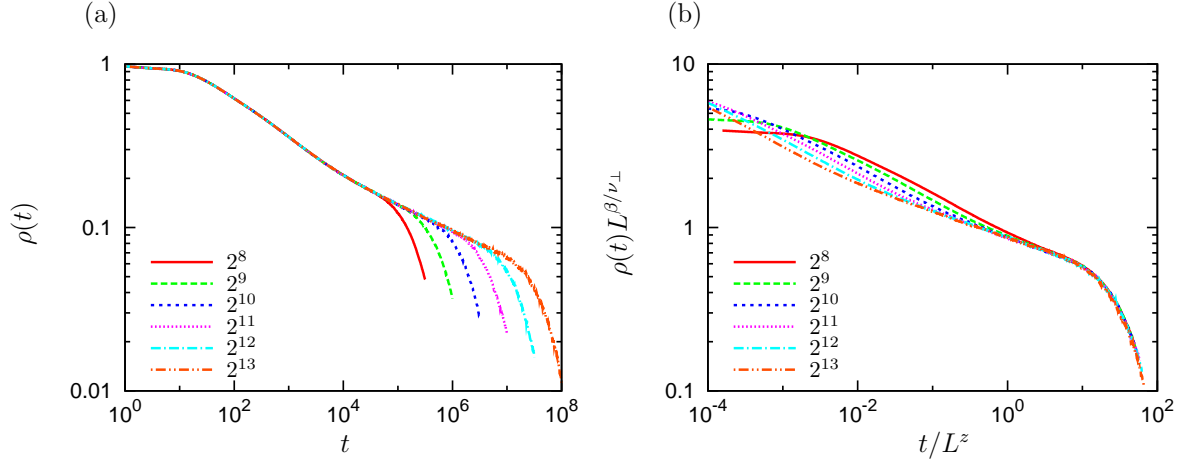
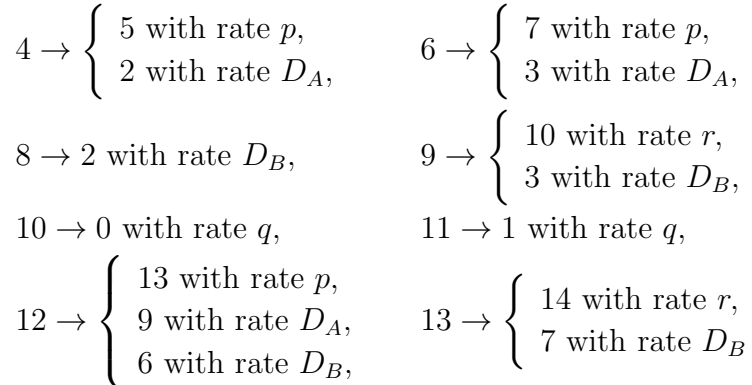


Figure 3. (a) Plots of $\rho_s(t)$ vs t at criticality ($q = 0.02$, $r = 0.9$, $p = 0.153\ 81$) for $L = 2^8, 2^9, 2^{10}, 2^{11}, 2^{12}$, and 2^{13} . (b) Plots of $\rho_s(t)L^{\beta/\nu_\perp}$ vs t/L^z at criticality with DP exponents $\beta/\nu_\perp \approx 0.252$ and $z \approx 1.58$. As anticipated from Figure 2(a), the data collapse becomes good when the time that the finite size effect appears is larger than 10^5 .

belong to the DP class. From the experience of the PCPD and the TCPD, we know that diffusion introduced to the model with IMAS may change the universality class. Quite interestingly, the exponents reported in Ref. [21] look very similar to those in Ref. [23] (in particular see the reported exponents for $r = 0.5$ in Table II of Ref. [23]), which studies a variant of the PCPD. Hence we investigated if Chatterjee et al. indeed observed the PCPD behavior from the 4SPP model with diffusion. Since the one-dimensional PCPD is known to have strong corrections to scaling and the true scaling behavior has not yet been properly observed, Ref. [21] might have observed the intermediated behavior of the PCPD, if the 4SPP model with diffusion is PCPD-like. To check this possibility, we study the role of biased diffusion, following the suggestion in Ref. [18]. That is, if the 4SPP model with biased diffusion shows the same critical behavior as the driven PCPD [18], one can conclude that the 4SPP with unbiased diffusion will share criticality with the PCPD.

Let D_A (D_B) be a transition rate with which A (B) moves to its right nearest neighbor. As reasoned above, movement of both species are assumed to be biased. For completeness, we fully write the transition rate including diffusion:



$$14 \rightarrow \begin{cases} 15 \text{ with rate } p, \\ 4 \text{ with rate } q, \\ 11 \text{ with rate } D_A, \end{cases} \quad 15 \rightarrow 5 \text{ with rate } q. \quad (7)$$

Now time-increment dt is chosen to be in the range

$$0 < dt \leq \frac{1}{\max(p + D_A + D_B, p + q + D_A, r + D_B)}. \quad (8)$$

In practice, we chose transition rates such that $dt = 1$ can be used near criticality.

If $D_A \neq 0$ and $D_B = 0$, the change of the universality class is not expected, because the model still has IMAS and no other symmetry and/or conservation emerges. The order parameter defined in Equation (4) is again used for $D_B = 0$ because the classification of active bonds does not change. Indeed, as we will see later, the diffusion of A does not affect the universal behavior.

The structure of the absorbing states is significantly affected by diffusion of B . If $D_B \neq 0$, an isolated predator can diffuse around until it meets another A or B . Hence the number of absorbing states shrinks to 3; the completely empty configuration, a configuration with a single predator without any prey, and the A fully occupied configuration without B .

At first sight, the diffusion of predators seems to have a similar role of the diffusion in the PCPD. That is, an isolated predator neither dies out nor branches another offspring until it meets another predator or prey. On the other hand, death and branching events are mediated by different local states, that is, death requires two predators but branching requires one predator and one prey, although in the PCPD both branching and annihilation occur by the same local configuration (that is, by a pair). Actually, whether branching and annihilation is mediated by the same local configuration or not seems to be an important factor. Unlike the PCPD and the TCPD, the universal behavior of models with hybrid rules with $m > n$ remains intact despite of diffusion [11, 24, 25]. In this regard, the 4SPP with diffusing predators is expected to belong to the DP class irrespective of whether the diffusion is biased or not, because this model is similar to the model with hybrid-rules of $m > n$ rather than the PCPD. We will show that the 4SPP model indeed belongs to the DP class.

Since the diffusion of A is not expected to affect the critical behavior, we study only two cases; one is for $D_A = 0.1$ and $D_B = 0$ and the other is for $D_A = 0$ and $D_B = 0.1$. q and r are fixed to be 0.1 and 0.9, respectively, and p is chosen to be the tuning parameter as before. As in the study of the non-diffusive 4SPP model, we study both the critical decay and the finite size scaling. The resulting analyses are summarized in Figure 4. For the model with $D_A = 0.1$ and $D_B = 0$ which has IMAS, the critical point is found to be $p_c = 0.598\ 30(5)$ by studying $\rho(t)t^\delta$ with the DP exponent $\delta \simeq 0.1594$. As expected, the critical density decay shows DP behavior. The finite size scaling collapse with the DP exponents again confirms that the 4SPP with $D_A = 0.1$ and $D_B = 0$ does belong to the DP class [see Figure 4(a)].

For the case of $D_A = 0$ and $D_B = 0.1$, a bond is active if a bond variable is an

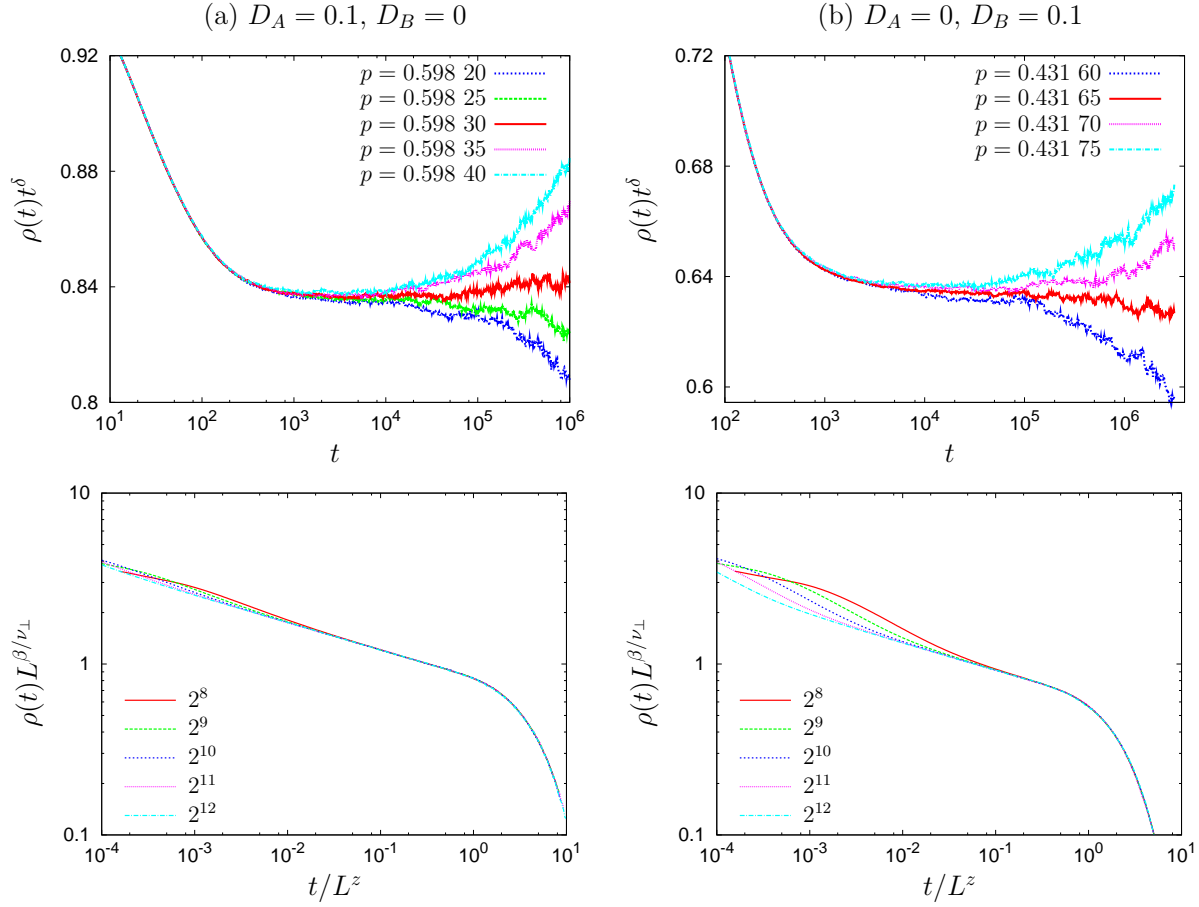


Figure 4. Numerical Study of critical phenomena for (a) $D_A = 0.1$ and $D_B = 0$ (two figures in left column) and for (b) $D_A = 0$ and $D_B = 0.1$ (two figures in right column). $q = 0.1$ and $r = 0.9$ is used. $\delta = 0.1594$, $z = 1.58$, and $\beta/\nu_\perp = 0.252$ (DP exponents) are used. From the two figures in above row, we conclude that $p_c = 0.592\ 30(5)$ for (a) and $p_c = 0.431\ 65(5)$ for (b). Two figures in lower row shows the finite size scaling collapse at corresponding criticality, which concludes that both models belong to the DP class.

element of the following set

$$S' = S \cup \{8\} = \{4, 6, 8, 9, 10, 11, 12, 13, 14, 15\}, \quad (9)$$

and now $N(t)$ is the number of these active bonds and the order parameter $\rho(t)$ is the density of such active bonds. In this case, the predator density also can be an order parameter. We again analyze the critical decay of the order parameter and the finite size scaling. At first, the density decay at $p_c = 0.431\ 65(5)$ is also well described by the DP exponent. The finite size scaling confirms that the 4SPP belongs to the DP class [see Figure 4(b)]. Hence, we conclude that irrespective of diffusion, the 4SPP does belong to the DP class.

To summarize, we studied the four-state predator-prey model introduced in Ref. [21] with and without diffusion. Unlike the conclusion made in Ref. [21], we found that the 4SPP does belong to the DP class regardless of whether prey and/or predators are

diffusive.

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